CE 297: Problems in the Mathematical Theory of Elasticity: Homework IV

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In the following, the Kolosov-Muskhelishvili potentials are represented using the symbols $\varphi(z), \Phi(z), \psi(z), \Psi(z), \chi(z)$*; the (real) Airy-stress function is represented using* $\phi(x, y)$

1. Use Goursat's method of factorizing the Laplacian ∇^2 to show that the biharmonic equation can be rewritten as
$$\frac{\partial^4 \phi}{\partial z^2 \partial \overline{z}^2} = 0$$
Formally integrate this expression (assuming the domain of interest is simply connected) to derive the basic K-M representation
$$2\phi = \overline{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}$$

2. Next, using the definition $\psi(z) \equiv \dfrac{d\chi}{dz}$, show that
$$\mathscr{F}(x, y) \equiv \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}$$

3. Recall that the physical interpretation of $\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}$ is that
$$F_x + iF_y = \int_{AB} t_x + it_y \, ds = -i\Big[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\Big]_A^B$$
where F_x, F_y are the components of the resultant force acting on an arc AB from the side of the outward-pointing normal and t_x, t_y are the traction components. Show

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similarly that the expression for resultant moment of the forces acting on AB about the origin is

$$M = \int_{AB} xt_y - yt_x \, ds = \mathrm{Re}\left[\chi(z) - z\psi(z) - z\overline{z}\,\varphi'(z)\right]_A^B$$

4. Consider an infinitesimal horizontal line segment to derive the complexified stress equation

$$\sigma_{yy} - i\,\sigma_{xy} = \varphi'(z) + \overline{\varphi'(z)} + z\,\overline{\varphi''(z)} + \overline{\psi'(z)}$$

5. Prove that the Type A substitutions to the K-M potentials

$$\begin{aligned} \varphi(z) &\to \varphi(z) + Ciz + \gamma \\ \psi(z) &\to \psi(z) + \gamma' \end{aligned}$$

where the constants $C \in \mathbb{R}$ and $\gamma, \gamma' \in \mathbb{C}$, leave the stresses invariant. What is their effect on the displacements?

6. For an infinite body with m holes show that the displacement for large z takes the form

$$2\mu(u + iv) = -\frac{\kappa\,(F_x + iF_y)}{2\pi(1+\kappa)} \log{(z\,\overline{z})} + \{(\kappa - 1)B + i(\kappa+1)C\}\,z - (B' - iC')\overline{z} + c_0$$

Hence derive the conditions under which the displacements are bounded at $\mathbb{C}_\infty$.

7. Recall that for an infinite elastic body with one hole enclosed by a simple closed contour L, the K-M potentials are

$$\begin{aligned} \varphi(z) &= -\frac{F_x + iF_y}{2\pi(1+\kappa)} \log z + \Gamma z + \varphi_o(z) \\ \psi(z) &= \frac{\kappa(F_x - iF_y)}{2\pi(1+\kappa)} \log z + \Gamma' z + \psi_o(z) \end{aligned}$$

where φ_o, ψ_o are functions which are holomorphic everywhere in the body including C_∞, and it is assumed that the origin lies inside the hole (without loss of generality). Show that the rigid-body rotation at infinity is

$$\omega_\infty = \frac{1+\kappa}{2\mu} C, \quad C = \mathrm{Im}\,(\Gamma)$$

8. Carry out the transformation procedure discussed in class for the K-M displacement

equation to show that the K-M stress equations in polar are as follows:

$$\begin{aligned}
\sigma_{rr} + \sigma_{\theta\theta} &= 4\,\mathrm{Re}\left\{\Phi(z)\right\} \\
\sigma_{\theta\theta} - \sigma_{rr} + 2i\,\sigma_{r\theta} &= 2e^{2i\theta}\left[\overline{z}\,\Phi'(z) + \Psi(z)\right] \\
\sigma_{rr} - i\,\sigma_{r\theta} &= \Phi(z) + \overline{\Phi(z)} - e^{2i\theta}\left[\overline{z}\,\Phi'(z) + \Psi(z)\right]
\end{aligned}$$

9. Show that if all three stress components are zero in any (open) domain in the interior of a planar elastic body, they are zero everywhere in the body. For simplicity, you can assume that the domain is a small open disk in the body. Can you arrive at the same conclusion given that the stresses vanish on a simple smooth contour lying inside the body (rather than an open domain)?

10. Consider a finite elastic body bounded by one simple, closed contour L. Recall that for the first (traction) boundary value problem, a solution exists only if the resultant of the external forces vanishes, i.e.

$$F_x + iF_y = \int_L t_x + it_y\,ds = -i\left[f_1 + if_2\right]_L = 0$$

Show that the condition for the vanishing of the resultant moments is

$$\int_L f_1\,dx + f_2\,dy = 0$$

11. Let $N(\theta)$, $T(\theta)$ be real, continuous, periodic functions of θ. Derive the complex Fourier series representation

$$N - iT = \sum_{k=-\infty}^{\infty} A_k e^{ik\theta} \qquad A_k = \frac{1}{2\pi}\int_0^{2\pi} (N - iT)e^{-ik\theta}d\theta$$

12. Consider a circular elastic disc of radius R with specified boundary functions $N(\theta)$ and $T(\theta)$, whose positive senses are specified such that $N(\theta) = \sigma_{rr}|_{r=R}, T(\theta) = \sigma_{r\theta}|_{r=R}$. Solve this first boundary value problem using appropriate series representations for $\Phi(z)$ and $\Psi(z)$, and explain the form of these potentials. Assume a complex Fourier representation for $N - iT$.

13. Using the result in Q12 above, find the stresses and displacements in a solid elastic cylinder of radius R subjected to (i) an external gas pressure p_0 (ii) a variable pressure $p_0 \sin^3\theta$.

14. Consider the problem of an infinite plate with a circular hole of radius R and a tensile remote stress $\sigma_{xx} = \sigma^{\infty}$. Use the potentials derived in class to find the stresses and displacements in the plate. Find the maximum value of the hoop stress $\sigma_{\theta\theta}$.

15. Find the stresses and displacements in the same plate when the applied remote stress is equibiaxial, i.e. $\sigma_{xx} = \sigma_{yy} = \sigma^{\infty}$.